EGG VOLUME

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Preston (1974, Auk 91: 132) has given equations for calculating the volume of an egg, assuming that the shape of an egg can be described by the revolution about its long axis of an oval figure whose parametric equations are

$$y = b \sin\theta \tag{1}$$

$$x = a \cos\theta\,(1 + c_1 \sin\theta + c_2 \sin^2\theta), \tag{2}$$

where c_1 and c_2 are coefficients representing the departure of the oval from an ellipse. In particular, c_1 represents a departure from symmetry, being zero for a symmetric egg.

Preston's analysis was in two parts. First he expressed the volume, V, in terms of the length, L, and the breadth, B, measured half way between the poles of the egg. Then he found a relation between the easily measured maximum breadth $B_{\max}$ and B.

A further step, which would be most immediately useful in practice, would be to combine the two parts of this analysis and to give an expression for V directly in terms of the measurable quantities L and $B_{\max}$.

Preston's formula for V was developed to first order in c_1 and c_2. (To this order it is independent of c_1.) His formula for $B_{\max}/B$ was developed to second order in c_1 and c_2. (To this order it is independent of c_2.) In order to express V directly in terms of L and $B_{\max}$, it is necessary that both parts of Preston's analysis should be carried to the same order. The analysis here is carried to the second order.

For the volume in terms of the length and equatorial breadth we have, to second order in c_1 and c_2:

$$V = \frac{\pi L B^2}{6}\left(1 + \tfrac{2}{5}c_2 + \tfrac{1}{5}c_1^2 + \tfrac{3}{35}c_2^2\right), \tag{3}$$

in agreement, to first order, with Preston's expression.

In retracing Preston's derivation of his expression for $B_{\max}/B$, I noted some small errors. Preston's equation (8) should read

$$3c_2 \sin^3\theta_m + 2c_1 \sin^2\theta_m + (1 - 2c_2)\sin\theta_m - c_1 = 0. \tag{4}$$

For small values of $\sin\theta_m$, c_1 and c_2, Preston finds $\sin\theta_m \simeq c_1$. This is actually correct to first order and follows from the above equation (4), although it would not follow from Preston's erroneous equation (8). However, in order to derive $B_{\max}/B$ correct to second order, $\sin\theta_m$ must also

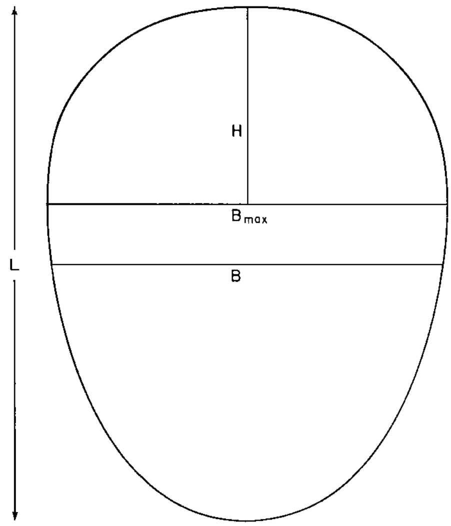

Fig. 1. An egg.

be taken to second order. The second order expression is

$$\sin \theta_m = c_1 + 2c_1c_2. \tag{5}$$

The resulting expression obtained for B_{max}/B is, to second order

$$B_{max}/B = 1 + \tfrac{1}{2}c_1^2. \tag{6}$$

The exact agreement between this and Preston's expression is fortuitous.

We can now proceed and combine equations (3) and (6) to express V directly in terms of L and B_{max}. To second order in c_1 and c_2 this is

$$V = \frac{\pi L B^2_{max}}{6}(1 + \tfrac{2}{5}c_2 - \tfrac{4}{5}c_1^2 + \tfrac{3}{35}c_2^2). \tag{7}$$

Having derived this equation, how then in practice is V to be determined? There will be four measurements to make: L, B, B_{max}, and H, as indicated in Fig. 1. The coefficient c_1 is then determined from equation (6). The value of $\sin \theta_m$ is given by

$$\sin \theta_m = 1 - 2H/L \tag{8}$$

so that c_2 can be determined from equation (5). The volume can then be calculated from equation (7) or read from Fig. 2, in which the number beside each hyperbola is $V \div (\pi L B^2_{max}/6)$.

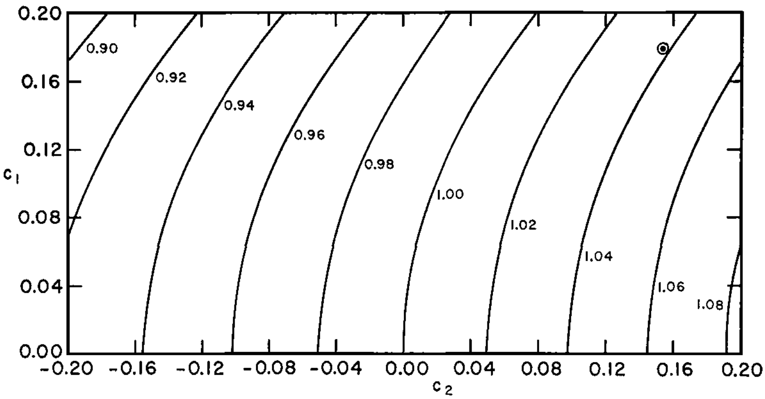

Fig. 2. Graphical representation of equation (7). The number beside each hyperbola is $6V/(\pi LB^2_{\max})$.

On the original drawing from which Fig. 1 was prepared, the measured quantities were

$$\begin{aligned} L &= 10.00 \text{ in} \\ B &= 7.50 \text{ in} \\ B_{\max} &= 7.62 \text{ in} \\ H &= 3.83 \text{ in.} \end{aligned}$$

From these, we find $\sin \theta_m = 0.234$, $c_1 = 0.179$, $c_2 = 0.154$, and hence $V = 1.038$ $(\pi LB^2_{\max}/6) = 315.6$ in^3.

The above analysis is adequate for slightly asymmetric eggs. However, following Preston, we have up to this point neglected the term in $\sin^3\theta_m$ in equation (4). This device is no longer adequate for a highly asymmetric egg. Indeed in the above example based on Fig. 1, the errors are just beginning to become appreciable. In fact, Fig. 1 was calculated with $c_1 = 0.180$, $c_2 = +0.150$, and the solution of equation (4) for these values is $\sin \theta_m = 0.2241$. For highly asymmetric eggs, what is required is a rapid numerical solution of the cubic equation (4).

A rough solution may be found very quickly by interpolation in Fig. 3, in which the number written against each line is the value of $\sin \theta_m$. For example, if $c_1 = 0.180$ and $c_2 = +0.150$, it is seen that $\sin \theta_m = 0.22$ approximately.

A more precise solution may be found fairly rapidly by the use of Fig. 4. If c_1 and c_2 are supposed known, $\sin \theta_m$ can be found as follows:

On Fig. 4 join the two points

$$\left(\frac{-3c_2}{2(1-2c_2)}, 0\right) \quad \text{and} \quad \left(0, -\frac{9c_2^2}{8c_1^2}\right)$$

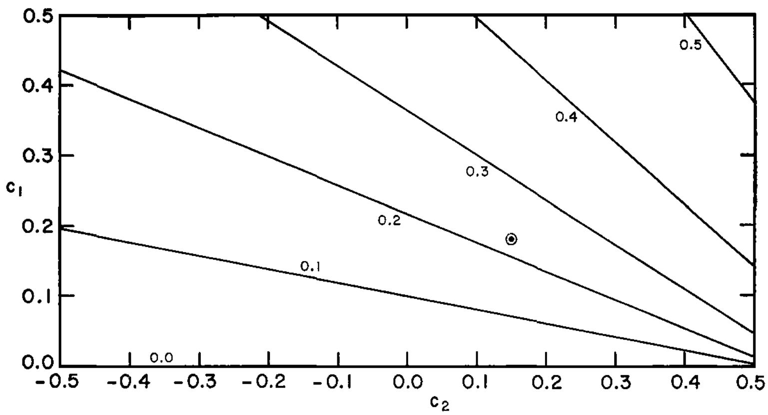

Fig. 3. Graph for quick approximate solution of equation (4). The number beside each line is $\sin\,\theta_m$.

with a straight line. Read off the value of x where this line intersects the curve. Then

$$\sin\,\theta_m = -\frac{2\,c_1 x}{3\,c_2}. \tag{9}$$

For example, suppose $c_1 = 0.180$ and $c_2 = +0.150$. The points to be joined are

$$(-0.321, 0) \qquad \text{and} \qquad (0, -0.781).$$

These intersect the curve at about $x = -0.28$, so that $\sin\,\theta_m = 0.224$ approximately. After this, a single refinement by a standard Newton-Raphson iterative procedure yields $\sin\,\theta_m = 0.2241$, which is correct to four significant figures.

It might be remarked that the exact shape of an asymmetric egg is a matter of considerable practical importance, for an account is given by Captain Lemuel Gulliver (quoted by Swift 1726, Voyage to Lilliput, London, Motte, ch. IV) of a war fought over this issue by the two great nations of Lilliput and Blefuscu in the 17th century:

> "His present Majesty's Grand-father, while he was a Boy, going to eat an Egg, and breaking it according to the ancient Practice, happened to cut one of his Fingers. Whereupon the Emperor his Father, published an Edict, commanding all his Subjects, upon great Penalties, to break the smaller End of their Eggs. The People so highly resented this Law, that our Histories tell us, there have been six Rebellions raised on that Account; wherein one Emperor lost his Life, and another his Crown. These civil Commotions were con-

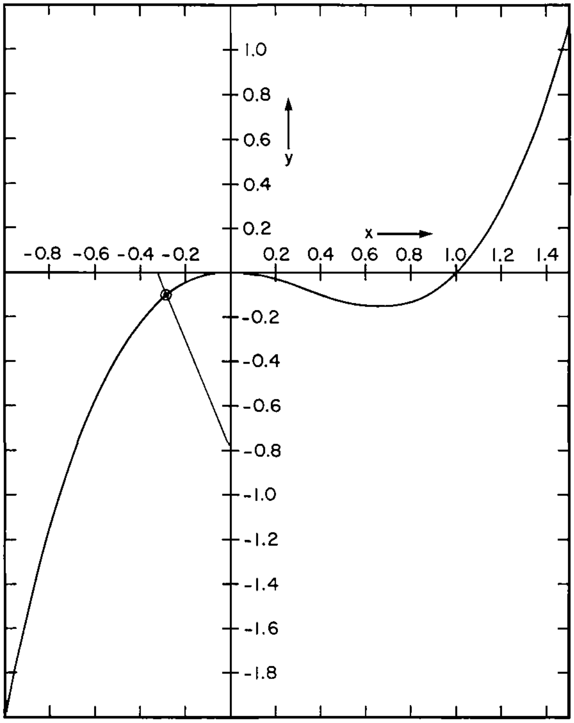

Fig. 4. Graph for more precise solution of equation (4).

stantly fomented by the Monarchs of *Blefuscu* . . . It is computed, that eleven Thousand Persons have, at several Times, suffered Death, rather than submit to break their Eggs at the smaller End. Many hundred large Volumes have been published upon this Controversy."

The print in these volumes is, however, rather small, and it is hoped that this contribution might help to avert a repetition of these calamities; any use that ornithologists might make of the theory is a welcome bonus.

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